Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 3, pp. 519–524, 2009

RADIAL EXPANSION OF A GRANULAR MEDIUM IN SPHERICAL AND CYLINDRICAL LAYERS

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UDC 539.37

An exact solution that describes the fields of displacements and stresses in an expanding spherical layer is constructed within the framework of the theory of small strains of a granular medium with rigid particles. For finite strains, the problem reduces to a nonlinear system of ordinary differential equations, which is solved by numerical methods. Similar solutions are found in the problem for a cylindrical layer. Based on these solutions, the effect of the dilatancy of the granular medium on the stress-strain state near expanding cavities is found.

Key words: *granular medium, elasticity, dilatancy, Hencky logarithmic strain tensor, variational inequality.*

Introduction. The simplest exact solution that describes a spherically symmetric state of an incompressible elastic medium around an expanding cavity is constructed by analogy with the solution of the problem of a point source in viscous fluid hydrodynamics [1]. This solution is independent of the cavity radius and has the form

$$u = \frac{Q}{4\pi r^2}, \qquad \sigma_r = -p - \frac{\mu Q}{\pi r^3}, \qquad \sigma_\varphi = \sigma_\psi = -p + \frac{\mu Q}{2\pi r^3}.$$

Here u is the radial displacement of the point, σ_r, σ_φ, and σ_ψ are the components of the stress tensor in a spherical coordinate system, p is the pressure at infinity, Q is the change in the cavity volume (mass flow), and μ is the shear modulus. The same distribution of displacements and stresses is formed in an expanding elastic layer with a given pressure applied to its outer boundary.

The solution of the problem with cylindrical symmetry has the form

$$u = \frac{Q}{2\pi r}, \qquad \sigma_r = -p - \frac{\mu Q}{2\pi r^2}, \qquad \sigma_\varphi = -p + \frac{\mu Q}{2\pi r^2}, \qquad \sigma_z = -p.$$

Deformation of a granular medium, in contrast to an elastic medium, involves a dilatational increase in volume owing to shear, and the deformation process becomes much more complicated, especially if the change in the cavity volume is rather large. In this case, the dilatancy occurs only until the state of ultimate loosening of the material is reached (in this state, the material becomes incompressible). Let us study the arising stress–strain state with both small and finite strains of the medium.

1. Expansion of a Spherical Layer. Let us assume that the inner surface of a spherical layer $r_0 < r < r_1$ of a densely packed granular medium containing rigid particles is expanding, its radial displacement is u_0, and a compressive stress equal to $-q$ is applied to the outer surface. For such a medium, the dilatancy equation [2] has the form

$$æ\gamma(\varepsilon) = \vartheta(\varepsilon), \tag{1.1}$$

where $æ \geq 0$ is the internal friction parameter, $\gamma(\varepsilon)$ is the shear intensity, $\vartheta(\varepsilon)$ is the volume strain

$$\gamma(\varepsilon) = \sqrt{\frac{2}{3}\sum_{i>j}(\varepsilon_i - \varepsilon_j)^2}, \qquad \vartheta(\varepsilon) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3;$$

and ε_i are the principal values of the small strain tensor ε.

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In a spherically symmetric case, the principal values are $\varepsilon_r = u'$ and $\varepsilon_\varphi = \varepsilon_\psi = u/r$ (the prime means the derivative with respect to r). Taking into account that $\varepsilon_\varphi > \varepsilon_r$ in the case of layer expansion, we obtain

$$\frac{2æ}{\sqrt{3}}\Big(\frac{u}{r} - u'\Big) = u' + 2\,\frac{u}{r}.$$

The solution of this equation satisfying the boundary condition at $r = r_0$ has the form

$$u = u_0\Big(\frac{r_0}{r}\Big)^\alpha, \qquad \alpha = 2\,\frac{\sqrt{3} - æ}{\sqrt{3} + 2æ}.$$

At $æ = 0$, the solution coincides with the above-given solution for an incompressible medium if u_0 is expressed via Q. At $æ < \sqrt{3}$, the displacement decreases with increasing r and tends to zero in an infinite layer. At $æ > \sqrt{3}$, the displacement increases, whereas the strains

$$\varepsilon_r = -\alpha\varepsilon_\varphi, \qquad \varepsilon_\varphi = \varepsilon_\psi = \frac{u_0}{r_0}\Big(\frac{r_0}{r}\Big)^{\alpha+1}$$

decrease independent of $æ$ and remain small at sufficiently small values of u_0. The condition $\varepsilon_r + 2\varepsilon_\varphi > 0$ is always satisfied; if this condition is violated, Eq. (1.1) makes no sense.

Note that another differential equation follows from Eq. (1.1) in the case of layer contraction ($u_0 < 0$) with $\varepsilon_r > 0$ and $\varepsilon_\varphi < 0$:

$$\frac{2æ}{\sqrt{3}}\Big(u' - \frac{u}{r}\Big) = u' + 2\,\frac{u}{r}.$$

The solution of this equation

$$u = u_0\Big(\frac{r_0}{r}\Big)^\beta, \qquad \beta = 2\,\frac{\sqrt{3} + æ}{\sqrt{3} - 2æ}$$

makes sense only if $æ < \sqrt{3}/2$, because the volume strain $\vartheta(\varepsilon) = \varepsilon_r + 2\varepsilon_\varphi$ becomes negative at greater values of $æ$. In this case, no solution exists for Eq. (1.1): the effect of particle jamming is observed; the medium is not deformed and remains in a rigid state.

To find the stresses, we write the constitutive relations for an ideal granular medium in the form of a variational inequality [2]

$$\sum_{i=1}^{3} \sigma_i(\tilde{\varepsilon}_i - \varepsilon_i) \le 0,$$

where $\tilde{\varepsilon}_i$ are the components of an arbitrary tensor $\tilde{\varepsilon}$ satisfying the constraint $æ\gamma(\tilde{\varepsilon}) \le \vartheta(\tilde{\varepsilon})$. With the use of the Kuhn–Tucker theorem, this inequality is transformed to the equations

$$\frac{\sigma_i}{p} = \frac{4æ}{3\gamma(\varepsilon)}\Big(\varepsilon_i - \frac{\varepsilon_j + \varepsilon_k}{2}\Big) - 1, \tag{1.2}$$

where $p > 0$ is the Lagrangian factor equal to the hydrostatic pressure and $i \ne j \ne k$. In the case of spherical symmetry, Eqs. (1.2) with allowance for Eq. (1.1) are transformed to

$$\sigma_r = -(2æ/\sqrt{3} + 1)p, \qquad \sigma_\varphi = \sigma_\psi = (æ/\sqrt{3} - 1)p.$$

From the equilibrium equation

$$\sigma_r' + 2(\sigma_r - \sigma_\varphi)/r = 0 \tag{1.3}$$

and the boundary condition $\sigma_r = -q$ on the outer surface, we find

$$p = \frac{\sqrt{3}\,q}{\sqrt{3} + 2æ}\Big(\frac{r_1}{r}\Big)^{2-\alpha}.$$

The stress on the inner surface of the layer is determined via the hydrostatic pressure calculated by this formula with $r = r_0$.

As the power index $2 - \alpha = 6æ/(\sqrt{3} + 2æ)$ is rigorously positive, the resultant solution makes no sense for an infinite layer ($r_1 \to \infty$) if the stress q differs from zero. In this case, expansion of the spherical cavity is impossible, because it requires an infinite stress to act from the side of the cavity on the medium.

At finite strains, when the displacement u_0 is not small, the strain state of the granular medium is described by the Hencky logarithmic tensor [3] with non-zero components $h_r = \ln R'$ and $h_\varphi = h_\psi = \ln(R/r)$, where

$R = r + u$ is the Eulerian coordinate of the particle. The dilatancy equation derived from Eq. (1.1) by means of replacing the small strain tensor with the logarithmic tensor is transformed to

$$R' = (r/R)^{\alpha}. \tag{1.4}$$

In contrast to the case of small strains, the parameter α calculated by the formula given above depends here on density:

$$\rho = \rho_0 \exp(-\vartheta(h)), \qquad \vartheta(h) = h_r + 2h_\varphi = \ln(R^2 R'/r^2).$$

For a medium with moderate dilatational expansion, such a dependence is approximately described by the expression

$$æ(\rho) = \begin{cases} æ_0 \Big(\dfrac{1 - \rho_*/\rho}{1 - \rho_*/\rho_0}\Big)^n, & \rho \geq \rho_*, \\ 0, & \rho < \rho_*, \end{cases}$$

where ρ_0 and ρ_* are the densities in the initial state and in the state of limiting dilatancy. Thus, Eq. (1.4) is a differential equation, which is not resolved with respect to the derivative. Direct calculations, however, show that

$$\frac{d\alpha}{dæ} < 0, \qquad \frac{dæ}{d\rho} \geq 0, \qquad \frac{d\rho}{dR'} < 0;$$

for $R \geq r$, therefore, the condition

$$1 - \frac{d}{dR'}\Big(\frac{r}{R}\Big)^{\alpha} = 1 - \Big(\frac{r}{R}\Big)^{\alpha} \ln\Big(\frac{r}{R}\Big) \frac{d\alpha}{dæ} \frac{dæ}{d\rho} \frac{d\rho}{dR'} \geq 1$$

is satisfied, which ensures problem solvability on the basis of the theorem of an implicit function.

By virtue of Eq. (1.4), we have

$$\frac{\rho}{\rho_0} = \frac{r^2}{R^2 R'} = \Big(\frac{r}{R}\Big)^{2-\alpha}.$$

Using this relation and assuming that $r = r_0$, we can express the radial displacement on the inner surface of the layer as a function of the medium density near this surface:

$$\frac{u_0}{r_0} = \Big(\frac{\rho_0}{\rho}\Big)^{1/(2-\alpha)} - 1.$$

The inverse dependence allows us to determine the density as a function of a given displacement. Typical plots of the inverse dependence obtained by numerical calculations are shown in Fig. 1. The coefficient of internal friction $æ_0$ in the state of dense packing in Fig. 1a is 0.5, and its value in Fig. 1b is $æ_0 = 2.5$. At $æ_0 = \sqrt{3}$ $(0.5 < æ_0 < 2.5)$, the parameter $\alpha(\rho_0)$ changes its sign. As was found for the case of small strains, this leads to a qualitative change in the field of displacements.

Figure 2 shows the results of the numerical solution of Eq. (1.4) by the Euler method with recalculation of the second order of accuracy after resolving this equation with respect to the derivative by the Newton–Raphson method. The Raphson correction turned out to be necessary for $n \leq 1$, where the usual Newton method does not converge because of the discontinuity of the derivative $dæ/d\rho$. An analysis of results shows that the medium dilatancy occurs mainly near the inner surface of the layer, and the limiting density of the material ρ_* is reached only as $u_0 \to \infty$.

The stresses in the layer can be found via strains by Eqs. (1.2) with the tensor ε being replaced by the tensor h:

$$\sigma_r = Ap, \qquad \sigma_\varphi = \sigma_\psi = Bp,$$

$$A = \frac{4æ^2}{3} f - 1, \qquad B = -\frac{2æ^2}{3} f - 1, \qquad f = \frac{h_r - h_\varphi}{h_r + 2h_\varphi} = \frac{\ln(rR'/R)}{\ln(R^2R'/r^2)}.$$

In these formulas, the hydrostatic pressure is the solution of the differential equation of equilibrium (1.3) in the Eulerian variables

$$\frac{d(Ap)}{dR} + \frac{2(A - B)p}{R} = 0,$$

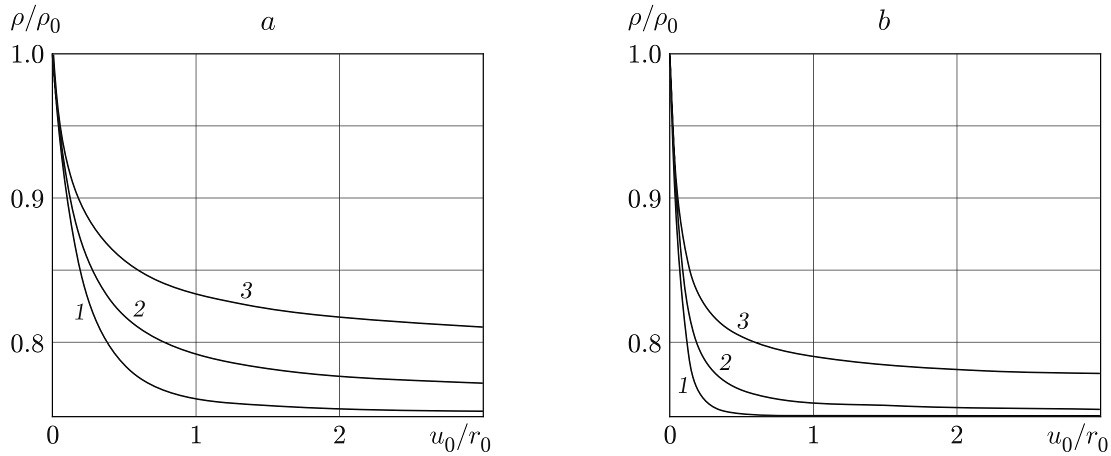

Fig. 1. Density of the medium near the inner surface versus the radial displacement ($\rho_* = 0.75\rho_0$) for $æ_0 = 0.5$ (a) and $= 2.5$ (b); $n = 0.5$ (1), 1 (2), and 2 (3).

which was solved numerically with allowance for the boundary condition on the outer surface of the layer by the Crank–Nicholson difference scheme [4]:

$$\frac{A_j p_j - A_{j-1} p_{j-1}}{R_j - R_{j-1}} + 2\,\frac{(A_j - B_j)p_j + (A_{j-1} - B_{j-1})p_{j-1}}{R_j + R_{j-1}} = 0.$$

The scheme allows the hydrostatic pressure in nodes to be calculated by explicit formulas, with subsequent recalculation of stresses. The thus-obtained distributions of σ_r and σ_φ inside the layer for $æ_0 = 2.5$ are plotted in Fig. 3. For $æ_0 = 0.5$, the corresponding curves are smoother: on curve 1, the absolute values of the stresses σ_r and σ_φ on the inner surface of the layer are approximately three times greater than the corresponding values on the outer surface.

The above-described calculation algorithms were tested through comparisons of the calculated results with exact solutions for small strains of the medium.

2. Expansion of a Cylindrical Layer. In a cylindrical layer, the principal values of the small strain tensor are $\varepsilon_r = u'$, $\varepsilon_\varphi = u/r$, and $\varepsilon_z = 0$; hence, Eq. (1.1) reduces to the nonlinear differential equation

$$\frac{2æ}{\sqrt{3}}\sqrt{(u')^2 - \frac{u'u}{r} + \frac{u^2}{r^2}} = u' + \frac{u}{r}, \qquad (2.1)$$

whose general solution $u = C/r^\alpha$ with the constant $C > 0$ and

$$\alpha = \frac{1 + 2æ^2/3 - 2æ\sqrt{1 - æ^2/3}}{1 - 4æ^2/3}$$

makes sense only if $æ \le \sqrt{3}$. For $æ > \sqrt{3}$, plane deformation of the granular medium is impossible, in contrast to axial expansion, because particle jamming occurs [2]. The value of α satisfies the condition $\alpha \le 1$, which ensures a non-negative right side of Eq. (2.1). The constant $C = u_0\, r_0^\alpha$ is determined by virtue of the boundary condition on the inner surface.

In accordance with Eqs. (1.2), the stresses in the layer are

$$\frac{\sigma_r}{p} = -\frac{2æ^2}{3}\,\frac{1 + 2\alpha}{1 - \alpha} - 1, \qquad \frac{\sigma_\varphi}{p} = \frac{2æ^2}{3}\,\frac{2 + \alpha}{1 - \alpha} - 1, \qquad \frac{\sigma_z}{p} = -\frac{2æ^2}{3} - 1.$$

Integrating the equilibrium equation

$$\sigma_r' + (\sigma_r - \sigma_\varphi)/r = 0,$$

and taking into account the boundary condition at $r = r_1$, we find

$$p = \frac{\beta q}{2æ\sqrt{1 - æ^2/3}}\Big(\frac{r_1}{r}\Big)^\beta, \qquad \beta = 2æ\,\frac{\sqrt{1 - æ^2/3} - æ}{1 - 4æ^2/3}. \qquad (2.2)$$

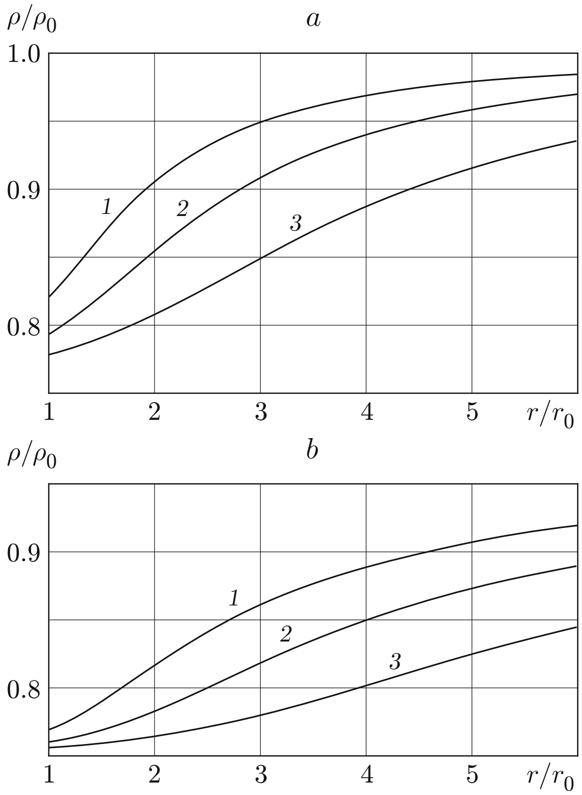

Fig. 2

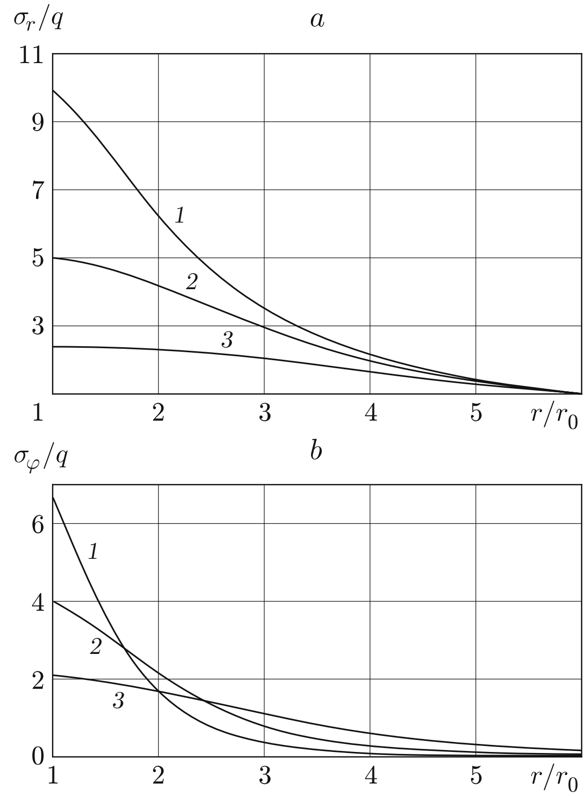

Fig. 3

Fig. 2. Distribution of density in a spherical layer ($n = 1$ and $\rho_* = 0.75\rho_0$) for $æ_0 = 0.5$ (a) and 2.5 (b); curves 1, 2, and 3 refer to $u_0/r_0 = 0.5$, 1, and 2, respectively.

Fig. 3. Distribution of stresses σ_r (a) and σ_φ (b) in a spherical layer ($æ_0 = 2.5$) for $u_0/r_0 = 0.5$ (1), 1 (2), and 2 (3).

The power index β monotonically increases from zero to two as the internal friction parameter æ changes from zero to $\sqrt{3}$. Therefore, the absolute values of the stresses in the layer always decrease along the radius.

For finite strains of the medium, the exact solution of the problem makes no sense. In this case, the displacements and stresses can be found by numerical integration of the first-order ordinary differential equations. With allowance for cylindrical symmetry, the principal values of the Hencky logarithmic tensor are $h_r = \ln R'$, $h_\varphi = \ln (R/r)$, and $h_z = 0$. The dilatancy equation (1.1) takes the form

$$æ\sqrt{\Big(\ln\frac{rR'}{R}\Big)^2 + (\ln R')^2 + \Big(\ln\frac{R}{r}\Big)^2} = \sqrt{\frac{3}{2}}\,\ln\frac{RR'}{r}.$$

Here the parameter æ depends on the medium density $\rho = \rho_0 r/(RR')$. To resolve this equation with respect to the derivative, one can apply, for instance, the Newton–Raphson method with subsequent integration by the Euler method.

The differential equation of equilibrium with respect to hydrostatic pressure is written in the form

$$\frac{d\,(Ap)}{dR} + \frac{(A-B)p}{R} = 0,$$

where

$$A = \frac{2æ^2}{3}\,\frac{\ln (r(R')^2/R)}{\ln (RR'/r)} - 1, \qquad B = -\frac{2æ^2}{3}\,\frac{\ln (r^2R'/R^2)}{\ln (RR'/r)} - 1$$

are functions used for determining the stresses: $\sigma_r = Ap$ and $\sigma_\varphi = Bp$. As in the case of a spherical layer, this equation can be solved numerically by using the Crank–Nicholson scheme. The calculation algorithm was tested on the basis of the exact solution of Eq. (2.2) within the framework of the theory of small strains.

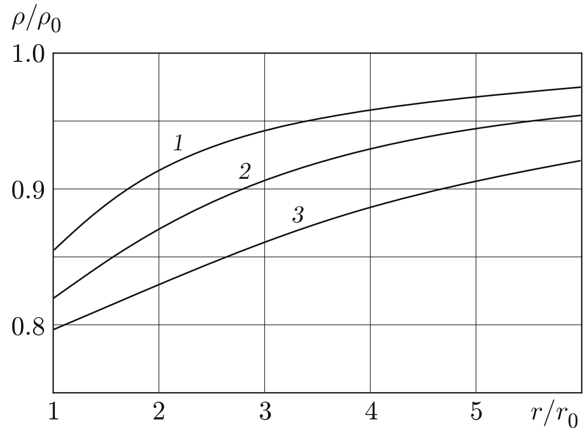

Fig. 4

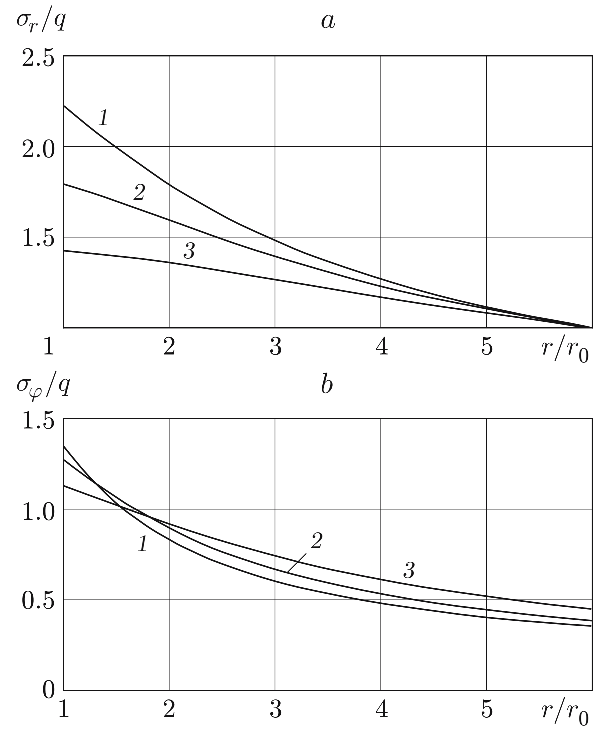

Fig. 5

Fig. 4. Distribution of density in a cylindrical layer ($æ_0 = 0.5$ and $n = 1$) for $u_0/r_0 = 0.5$ (1), 1 (2), and 2 (3).

Fig. 5. Distribution of stresses σ_r (a) and σ_φ (b) in a cylindrical layer ($æ_0 = 0.5$ and $n = 1$) for $u_0/r_0 = 0.5$ (1), 1 (2), and 2 (3).

A typical distribution of density in the cylindrical layer, which was obtained in calculations with $æ_0 = 0.5$ and $n = 1$, is plotted in Fig. 4. Figure 5 shows the distribution of the stresses σ_r and σ_φ inside the layer for the same values of the problem parameters as in Fig. 4. It follows from the analysis of the results obtained that the process of medium dilatancy accompanied by relaxation of shear stresses (transition of the stressed state of the medium to the hydrostatic state) in the cylindrical layer occurs slower than in the spherical layer, as the cavity radius is changed by the same value. This is caused by the constraint of the degree of freedom of particle motion in the z direction; for this reason, the shear intensity in the case of cylindrical symmetry is always lower than in the case of spherically symmetric motion.

To conclude, we should note that the exact and approximate solutions obtained can be used for testing algorithms of numerical implementation of mathematical models of mechanics of granular media.

This work was supported by the Russian Foundation for Basic Research (Grant No. 08-01-00148) and by the Complex Program of Basic Research of the Presidium of the Russian Academy of Sciences No. 14 “Fundamental Problems of Informatics and Informational Technologies.”

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